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Inverse Problem for a General Bounded Domain in R^3 with Piecewise Smooth Mixed Boundary Conditions

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We study the influence of a finite container on an ideal gas. The trace of the heat kernel $\Theta(t) = \sum_{\nu=1}^{\infty} \exp(-t\mu_\nu)$, where $\{\mu_\nu\}_{\nu=1}^{\infty}$ are the eigenvalues of the negative Laplacian $-\nabla^2 = -\sum_{\beta=1}^{3} (\partial/\partial x^\beta)^2$ in the (x^1, x^2, x^3)-space, is studied for a general bounded domain Ω with a smooth bounding surface S, where a finite number of Dirichlet, Neumann, and Robin boundary conditions on the piecewise smooth parts S_i $(i = 1, \ldots, n)$ of S are considered such that $S = \cup_{i=1}^{n} S_i$. Some geometrical properties of Ω (the volume, the surface area, the mean curvature, and the Gaussian curvature) are determined. Furthermore, thermodynamic quantities, particularly the energy, for an ideal gas enclosed in the general bounded domain Ω with Dirichlet, Neumann, and Robin conditions are examined with the help of the asymptotic expansions of $\Theta(t)$ for short time t. We show that these thermodynamic quantities depend on some geometric properties of Ω.

1. INTRODUCTION

Let $\Omega \subseteq R^3$ be a simply connected, bounded domain with a smooth bounding surface S. Consider the Robin problem

$$-\nabla^2 u = \mu u \qquad \text{in } \Omega \tag{1.1}$$

$$\left(\frac{\partial}{\partial n} + \gamma\right) u = 0 \qquad \text{on } S \tag{1.2}$$

where $\partial/\partial n$ denotes differentiation along the inward-pointing normal to S with $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Denote its eigenvalues, counted according to the multiplicity, by

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$$0 < \mu_1 \leq \mu_2 \leq \mu_3 \leq \ldots \leq \mu_\nu \leq \ldots \to \infty \qquad \text{as} \quad \nu \to \infty \tag{1.3}$$

The Robin problem (1.1)–(1.2) has been discussed by Zayed [11] when γ is a positive constant and by Zayed [17] when γ is a smooth function which is not strictly positive; geometric quantities associated with the bounded domain Ω have been determined, using the asymptotic expansions of the trace of the heat kernel

$$\Theta(t) = \sum_{\nu=1}^{\infty} \exp(-t\mu_\nu) \qquad \text{as} \quad t \to 0^+ \tag{1.4}$$

The Robin problem (1.1)–(1.2) has been investigated by many authors (see, for example, refs. 2, 4–6, 8, 10) in the following special cases:

Case 1. $\gamma = 0$ (*the Neumann problem*):

$$\Theta(t) = \frac{V}{(4\pi t)^{3/2}} + \frac{|S|}{16\pi t} + \frac{1}{12\pi^{3/2}t^{1/2}} \int_S H\, dS + \frac{7}{128\pi} \int_S (H^2 - N)\, ds + O(t^{1/2}) \quad \text{as} \quad t \to 0^+ \tag{1.5}$$

Case 2. $\gamma \to \infty$ (*the Dirichlet problem*):

$$\Theta(t) = \frac{V}{(4\pi t)^{3/2}} - \frac{|S|}{16\pi t} + \frac{1}{12\pi^{3/2}t^{1/2}} \int_S H\, dS + \frac{1}{128\pi} \int_S (H^2 - N)\, dS + O(t^{1/2}) \quad \text{as} \quad t \to 0^+ \tag{1.6}$$

In these formulas, V, $|S|$, H, and N are respectively the volume, the surface area, the mean curvature, and the Gaussian curvature of Ω, where $H = \frac{1}{2}(1/R_1 + 1/R_2)$ and $N = 1/R_1R_2$, while R_1 and R_2 are the principal radii of curvature.

Case 3 (the mixed problem). Let $|S_1|$, H, and N respectively be the surface area, mean curvature, and Gaussian curvature of a part S_1 of the surface S with the Neumann boundary conditions and let $|S_2|$, H, and N respectively be the surface area, mean curvature, and Gaussian curvature of the remaining part $S_2 = S \backslash S_1$ of S with the Dirichlet boundary conditions. Then, considering refs. 12–16, we obtain

$$\Theta(t) = \frac{V}{(4\pi t)^{3/2}} + \frac{|S_1| - |S_2|}{16\pi t} + \frac{1}{12\pi^{3/2}t^{1/2}} \left\{ \int_{S_1} H\, dS_1 + \int_{S_2} H\, dS_2 \right\} + \frac{1}{128\pi} \left\{ 7 \int_{S_1} (H^2 - N)\, dS_1 + \int_{S_2} (H^2 - N)\, dS_2 \right\}$$

$$+\left(\frac{t}{\pi^3}\right)^{1/2}\left\{\frac{13}{1440}\int_{S_1} H^3\, dS_1 - \frac{1}{315}\int_{S_2} H^3\, dS_2\right\} + O(t)$$
$$\text{as} \quad t \to 0^+ \tag{1.7}$$

The object of this paper is to discuss the following, more general inverse problem: Suppose that the eigenvalues (1.3) are given for the Helmholtz equation (1.1) together with the following Dirichlet, Neumann, and Robin boundary conditions:

$$\begin{aligned} u &= 0 \qquad \text{on} \quad S_i \quad (i = 1, \ldots, k) \\ \frac{\partial u}{\partial n_i} &= 0 \qquad \text{on} \quad S_i \quad (i = k+1, \ldots, m) \\ \left(\frac{\partial}{\partial n_i} + \gamma_i\right) u &= 0 \qquad \text{on} \quad S_i \quad (i = m+1, \ldots, n) \end{aligned} \tag{1.8}$$

where the bounding surface S of Ω consists of a finite number of piecewise smooth parts S_i $(i = 1, \ldots, n)$ such that $S = \cup_{i=1}^{n} S_i$ and $\partial/\partial n_i$ denotes differentiation along the inward-pointing normal to the parts S_i; the γ_i are positive constants.

The basic problem is to determine some geometric quantities associated with the problem (1.1) and (1.8) by using the asymptotic expansions of the trace of the heat kernel (1.4).

Note that the special cases of the main problem (1.1) and (1.8) have been discussed by many authors [e.g., 8, 10, 14–16]. Therefore, this problem can be considered as a more general one which does not seem to have been investigated elsewhere.

2. STATEMENT OF RESULTS

Suppose that the bounding surface S of the domain Ω is given locally by infinitely differentiable functions $x^\alpha = y^\alpha(\sigma)$ $(\alpha = 1, 2, 3)$ of the parameters σ^i $(i = 1, 2)$. If these parameters are chosen so that $\alpha^i =$ const are lines of curvature, the first and second fundamental forms of S can be written in the form

$$\Pi_1(\sigma, \Delta\sigma) = \sum_{i=1}^{2} g_{ii}(\sigma)(\Delta\sigma^i)^2$$

$$\Pi_2(\sigma, \Delta\sigma) = \sum_{i=1}^{2} d_{ii}(\sigma)(\Delta\sigma^i)^2$$

In terms of the coefficients g_{11}, g_{22}, d_{11}, and d_{22} the principal radii of curvature are $R_1 = g_{11}/d_{11}$, and $R_2 = g_{22}/d_{22}$. Let $|S_i|$ $(i = 1, \ldots, n)$ be the surface areas of the parts S_i $(i = 1, \ldots, n)$ of the surface S, respectively. Let $h_i > 0$ $(i = 1, \ldots, n)$ be sufficiently small. Let n_i $(i = 1, \ldots, n)$ be the minimum distances from a point $\mathbf{x} = (x^1, x^2, x^3)$ of the domain Ω to the parts S_i $(i = 1, \ldots, n)$, respectively. Let $\mathbf{n}_i\,(\sigma)$ $(i = 1, \ldots, n)$ denote the inward-drawn unit normal to the parts S_i $(i = 1, \ldots, n)$, respectively. Then, we note that the coordinates in the neighborhood of the parts S_i $(i = 1, \ldots, n)$ are of the same form as in Section 3 of Zayed [11] with the interchanges $n \leftrightarrow n_i$, $h \leftrightarrow h_i$, $I \leftrightarrow I_i$, $C(I) \leftrightarrow D(I_i)$, and $\delta^* \leftrightarrow \delta_i$. Thus, we have the same formulas (3.1)–(3.4) as in Section 3 of Zayed [11] with the interchanges $n \leftrightarrow n_i$, $n(\sigma) \to \mathbf{n}_i\,(\sigma)$, and $dS \leftrightarrow dS_i$.

Theorem 2.1. With the assumptions stated above, the asymptotic expansion of $\Theta(t)$ for small time t of the problem (1.1) and (1.8) can be written in the form

$$\Theta(t) = \frac{a_1}{t^{3/2}} + \frac{a_2}{t} + \frac{a_3}{t^{1/2}} + a_4 + a_5 t^{1/2} + O(t) \qquad \text{as} \quad t \to 0^+ \tag{2.1}$$

where, if $0 < \gamma_i << 1$ $(i = m + 1, \ldots, c)$ and $\gamma_i >> 1$ $(i = c + 1, \ldots, n)$, the coefficients a_ν $(\nu = 1\text{–}5)$ can be written in the forms

$$a_1 = \frac{V}{8\pi^{3/2}}$$

$$a_2 = \frac{1}{16\pi}\left\{\left[\sum_{i=k+1}^{m} |S_i| + \sum_{i=m+1}^{c} |S_i|\right] - \left[\sum_{i=1}^{k}|S_i| + \sum_{i=c+1}^{n}\left(|S_i| - 2\gamma_i^{-1}\int_{S_i} H\,dS_i\right)\right]\right\}$$

$$a_3 = \frac{1}{12\pi^{3/2}}\left\{\sum_{i=1}^{k}\int_{S_i} H\,dS_i + \sum_{i=k+1}^{m}\int_{S_i} H\,dS_i + \sum_{i=m+1}^{c}\int_{S_i}(H - 3\gamma_i)\,dS_i + \sum_{i=c+1}^{n}\int_{S_i} H\,dS_i\right\}$$

$$a_4 = \frac{1}{128\pi}\left\{\sum_{i=1}^{k}\int_{S_i}(H^2 - N)\,dS_i + 7\sum_{i=k+1}^{m}\int_{S_i}(H^2 - N)\,dS_i + 7\sum_{i=m+1}^{c}\int_{S_i}\left[(H - 3\gamma_i)^2 - \left(N - \frac{26}{7}\gamma_i H + \frac{47}{7}\gamma_i^2\right)\right] dS_i\right.$$

$$+ \sum_{i=c+1}^{n} \int_{S_i} [H^2 - (N - 16\gamma_i^{-1}H)] \, dS_i \Bigg\}$$

$$a_5 = \frac{1}{\pi^{3/2}} \Bigg\{ -\frac{1}{315} \sum_{i=1}^{k} \int_{S_i} H^3 \, dS_i + \frac{13}{1440} \sum_{i=k+1}^{m} \int_{S_i} H^3 \, dS_i$$

$$+ \frac{13}{1440} \sum_{i=m+1}^{c} \int_{S_i} (H - 3\gamma_i)^3 \, dS_i - \frac{1}{315} \sum_{i=c+1}^{n} \int_{S_i} H^3 \, dS_i \Bigg\}$$

From this theorem, we deduce the following corollaries:

Corollary 2.1. If we consider the problem (1.1) and (1.8) with $\gamma_i >> 1$ $(i = m + 1, \ldots, n)$, then the asymptotic expansion of $\Theta(t)$ follows directly from Theorem 2.1 by setting $c = m$ with $\Sigma_{i=m+1}^{m}$ as zero.

Corollary 2.2. If we consider the problem (1.1) and (1.8) with $0 < \gamma_i << 1$ $(i = m + 1, \ldots, n)$, then the asymptotic expansion of $\Theta(t)$ follows directly from Theorem 2.1 by setting $c = n$ with $\Sigma_{i=n+1}^{n}$ as zero.

Finally, let us add that the question raised by Kac [5], namely, "Can one hear the shape of a drum?" was answered negatively by Gordon *et al.* [1], who showed explicitly two domains that, although they have different shapes, they have the same eigenvalues (i.e., isospectral domains). This theoretical result was experimentally verified recently by employing thin microwave cavities shaped in the form of two different isospectral domains [9]. Milnor [7] had already showed in 1964 an example of two different domains with the same eigenvalues, but they were two 16-dimensional tori.

3. DERIVATION OF THE RESULTS

In analogy with the two-dimensional problem [18] and considering refs. 8, 10, and 11, it is easy to show that the trace of the heat kernel $\Theta(t)$ associated with the problem (1.1) and (1.8) is given by

$$\Theta(t) = \iiint_{\Omega} G(\mathbf{x}, \mathbf{x}; t) \, d\mathbf{x} \tag{3.1}$$

where $G(\mathbf{x}_1, \mathbf{x}_2; t)$ is the Green's function for the heat equation

$$\nabla^2 u = \frac{\partial u}{\partial t} \tag{3.2}$$

subject to the Dirichlet, Neumann, and Robin boundary conditions (1.8) and the initial condition

$$\lim_{t\to 0} G(\mathbf{x}_1, \mathbf{x}_2; t) = \delta\,(\mathbf{x}_1 - \mathbf{x}_2) \tag{3.3}$$

where $\delta(\mathbf{x}_1 - \mathbf{x}_2)$ is the Dirac delta function located at the source point $\mathbf{x}_2$. Let us write

$$G(\mathbf{x}_1, \mathbf{x}_2; t) = G_0(\mathbf{x}_1, \mathbf{x}_2; t) + \chi(\mathbf{x}_1, \mathbf{x}_2; t) \tag{3.4}$$

where

$$G_0(\mathbf{x}_1, \mathbf{x}_2; \mathbf{t}) = (4\pi t)^{-3/2} \exp\left\{-\frac{|\mathbf{x}_1 - \mathbf{x}_2|^2}{4t}\right\} \tag{3.5}$$

is the "fundamental solution" of the heat equation (3.2), while $\chi(\mathbf{x}_1, \mathbf{x}_2; t)$ is the "regular solution" chosen in such a way that $G(\mathbf{x}_1, \mathbf{x}_2; t)$ satisfies the Dirichlet, Neumann, and Robin boundary conditions (1.8). On setting $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x}$, we find that

$$\Theta(t) = \frac{V}{(4\pi t)^{3/2}} + R(t) \tag{3.6}$$

where

$$R(t) = \iiint_{\Omega} \chi\,(\mathbf{x}, \mathbf{x}, t)\, d\mathbf{x} \tag{3.7}$$

The problem now is to determine the asymptotic expansion of $R(t)$ as $t \to 0^+$. In what follows, we shall use Laplace transforms with respect to t, and use s^2 as the Laplace transform parameter; thus we define

$$\overline{G}(\mathbf{x}_1, \mathbf{x}_2; s^2) = \int_0^\infty e^{-s^2 t}\, G(\mathbf{x}_1, \mathbf{x}_2; t)\, dt \tag{3.8}$$

An application of the Laplace transform to the heat equation (3.2) shows that $\overline{G}\,(\mathbf{x}_1, \mathbf{x}_2; s^2)$ satisfies the membrane equation

$$(\nabla^2 - s^2)\overline{G}(\mathbf{x}_1, \mathbf{x}_2; s^2) = -\delta(\mathbf{x}_1 - \mathbf{x}_2) \qquad \text{in} \quad \Omega \tag{3.9}$$

together with the Dirichlet, Neumann, and Robin boundary conditions (1.8).

The asymptotic expansion of $R(t)$ as $t \to 0^+$ may then be deduced directly from the asymptotic expansion of $\overline{R}(s^2)$ as $s \to \infty$, where

$$\overline{R}(s^2) = \iiint_{\Omega} \overline{\chi}(\mathbf{x}, \mathbf{x}; s^2)\, d\mathbf{x} \tag{3.10}$$

It is well known [e.g., 8, 10, 11] that the membrane equation (3.9) has the fundamental solution

$$\overline{G}_0\,(\mathbf{x}_1, \mathbf{x}_2; s^2) = \exp(-sr_{\mathbf{x}_1\mathbf{x}_2})/4\pi r_{\mathbf{x}_1\mathbf{x}_2} \tag{3.11}$$

where $r_{\mathbf{x}_1\mathbf{x}_2} = |\mathbf{x}_1 - \mathbf{x}_2|$ is the distance between the points $\mathbf{x}_1 = (x_1^1, x_1^2, x_1^3)$ and $\mathbf{x}_2 = (x_2^1, x_2^2, x_2^3)$ of the domain Ω. The existence of such a solution enables us to construct integral equations for $\overline{G}(\mathbf{x}_1, \mathbf{x}_2; s^2)$ satisfying the Dirichlet, Neumann, and Robin boundary conditions (1.8) with small/large impedances γ_i. Therefore, if we consider the problem (1.1) and (1.8) with $0 < \gamma_i << 1$ $(i = m + 1, \ldots, c)$ and $\gamma_i >> 1$ $(i = c + 1, \ldots, n)$, then Green's theorem gives the following integral equation:

$$\begin{aligned}
\overline{G}(\mathbf{x}_1, \mathbf{x}_2; s^2) &= \exp(-sr_{\mathbf{x}_1\mathbf{x}_2})/4\pi r_{\mathbf{x}_1\mathbf{x}_2} \\
&+ \frac{1}{2\pi}\sum_{i=1}^{k}\int_{S_i}\left[\frac{\partial}{\partial n_{i\mathbf{y}}}\overline{G}\,(\mathbf{x}_1, \mathbf{y}; s^2)\right]\{\exp(-sr_{\mathbf{y}\mathbf{x}_2})/r_{\mathbf{y}\mathbf{x}_2}\}\, d\mathbf{y} \\
&- \frac{1}{2\pi}\sum_{i=k+1}^{m}\int_{S_i}[\overline{G}\,(\mathbf{x}_1, \mathbf{y}; s^2)]\frac{\partial}{\partial n_{i\mathbf{y}}}[\exp(-sr_{\mathbf{y}\mathbf{x}_2})/r_{\mathbf{y}\mathbf{x}_2}]\, d\mathbf{y} \\
&- \frac{1}{2\pi}\sum_{i=m+1}^{c}\int_{S_i}[\overline{G}\,(\mathbf{x}_1; \mathbf{y}, s^2)] \\
&\times\left[\left(\frac{\partial}{\partial n_{i\mathbf{y}}} + \gamma_i\right)\{\exp\,(-sr_{\mathbf{y}\mathbf{x}_2})/r_{\mathbf{y}\mathbf{x}_2}\}\right] d\mathbf{y} \\
&+ \frac{1}{2\pi}\sum_{i=c+1}^{n}\int_{S_i}\left[\frac{\partial}{\partial n_{i\mathbf{y}}}\overline{G}\,(\mathbf{x}_1, \mathbf{y}; s^2)\right] \\
&\times\left[\left(1 + \gamma_i^{-1}\frac{\partial}{\partial n_{i\mathbf{y}}}\right)\{\exp(-sr_{\mathbf{y}\mathbf{x}_2})/r_{\mathbf{y}\mathbf{x}_2}\}\right] d\mathbf{y}
\end{aligned} \tag{3.12}$$

On applying iteration methods [e.g., 11, 13, 14] to the integral equation (3.12), we obtain the Green's function $\overline{G}\,(\mathbf{x}_1, \mathbf{x}_2; s^2)$ which has a regular part in the following form:

$$\bar{\chi}(\mathbf{x}_1, \mathbf{x}_2; s^2) = \left(\sum_{i=1}^{20} A_i\right)\Big/ 8\pi^2 \tag{3.13}$$

where

$$A_1 = \sum_{i=1}^{k} \int_{S_i} \frac{\partial}{\partial n_{i\mathbf{y}}} [\exp(-sr_{\mathbf{x}_1\mathbf{y}})/r_{\mathbf{x}_1\mathbf{y}}] \{\exp(-sr_{\mathbf{y}\mathbf{x}_2})/r_{\mathbf{y}\mathbf{x}_2}\}\, d\mathbf{y}$$

$$A_2 = -\sum_{i=k+1}^{m} \int_{S_i} [\exp(-sr_{\mathbf{x}_1\mathbf{y}})/r_{\mathbf{x}_1\mathbf{y}}] \frac{\partial}{\partial n_{i\mathbf{y}}} [\exp(-sr_{\mathbf{y}\mathbf{x}_2})/r_{\mathbf{y}\mathbf{x}_2}]\, d\mathbf{y}$$

$$A_3 = -\sum_{i=m+1}^{c} \int_{S_i} [\exp(-sr_{\mathbf{x}_1\mathbf{y}})/r_{\mathbf{x}_1\mathbf{y}}] \left\{\left(\frac{\partial}{\partial n_{i\mathbf{y}}} + \gamma_i\right)[\exp(-sr_{\mathbf{y}\mathbf{x}_2})/r_{\mathbf{y}\mathbf{x}_2}]\right\} d\mathbf{y}$$

$$A_4 = \sum_{i=c+1}^{n} \int_{S_i} \frac{\partial}{\partial n_{i\mathbf{y}}}[\exp(-sr_{\mathbf{x}_1\mathbf{y}})/r_{\mathbf{x}_1\mathbf{y}}]\left\{\left(1 + \gamma_i^{-1} \frac{\partial}{\partial n_{i\mathbf{y}}}\right) [\exp(-sr_{\mathbf{y}\mathbf{x}_2})/r_{\mathbf{y}\mathbf{x}_2}]\right\} d\mathbf{y}$$

$$A_5 = \sum_{i=1}^{k} \int_{S_i}\int_{S_i} \frac{\partial}{\partial n_{i\mathbf{y}}} [\exp(-sr_{\mathbf{x}_1\mathbf{y}})/r_{\mathbf{x}_1\mathbf{y}}]\, M_{1i}(\mathbf{y}, \mathbf{y}')[\exp(-sr_{\mathbf{y}'\mathbf{x}_2})/r_{\mathbf{y}'\mathbf{x}_2}]\, d\mathbf{y}\, d\mathbf{y}'$$

$$A_6 = \sum_{i=k+1}^{m} \int_{S_i}\int_{S_i} [\exp(-sr_{\mathbf{x}_1\mathbf{y}})/r_{\mathbf{x}_1\mathbf{y}}]\, M_{2i}(\mathbf{y}, \mathbf{y}') \frac{\partial}{\partial n_{i\mathbf{y}'}} [\exp(-sr_{\mathbf{y}'\mathbf{x}_2})/r_{\mathbf{y}'\mathbf{x}_2}]\, d\mathbf{y}\, d\mathbf{y}'$$

$$A_7 = \sum_{i=m+1}^{c} \int_{S_i}\int_{S_i} [\exp(-sr_{\mathbf{x}_1\mathbf{y}})/r_{\mathbf{x}_1\mathbf{y}}]\, L_{\gamma_i}(\mathbf{y}, \mathbf{y}')$$
$$\times \left\{\left(\frac{\partial}{\partial n_{i\mathbf{y}'}} + \gamma_i\right) [\exp(-sr_{\mathbf{y}'\mathbf{x}_2})/r_{\mathbf{y}'\mathbf{x}_2}]\right\} d\mathbf{y}\, d\mathbf{y}'$$

$$A_8 = \sum_{i=c+1}^{n} \int_{S_i}\int_{S_i} \frac{\partial}{\partial n_{i\mathbf{y}}} [\exp(-sr_{\mathbf{x}_1\mathbf{y}})/r_{\mathbf{x}_1\mathbf{y}}]\, L_{\gamma_i^{-1}}(\mathbf{y}, \mathbf{y}')$$
$$\times \left\{\left(1 + \gamma_i^{-1} \frac{\partial}{\partial n_{i\mathbf{y}'}}\right) [\exp(-sr_{\mathbf{y}'\mathbf{x}_2})/r_{\mathbf{y}'\mathbf{x}_2}]\right\} d\mathbf{y}\, d\mathbf{y}'$$

$$A_9 = -\sum_{i=1}^{k} \int_{S_i} \left\{\sum_{i=k+1}^{m} \int_{S_i} [\exp(-sr_{\mathbf{x}_1\mathbf{y}})/ r_{\mathbf{x}_1\mathbf{y}}] M_{3i}(\mathbf{y}, \mathbf{y}')\, d\mathbf{y}\right\}$$
$$\times [\exp(-sr_{\mathbf{y}'\mathbf{x}_2})/r_{\mathbf{y}'\mathbf{x}_2}]\, d\mathbf{y}'$$

$$A_{10} = -\sum_{i=k+1}^{m} \int_{S_i} \left\{ \sum_{i=1}^{k} \int_{S_i} \frac{\partial}{\partial n_{i\mathbf{y}}} [\exp(-sr_{\mathbf{x}_1\mathbf{y}})/\, r_{\mathbf{x}_1\mathbf{y}}] M_{4i}\,(\mathbf{y}, \mathbf{y}')\, d\mathbf{y} \right\}$$

$$\times \frac{\partial}{\partial n_{i\mathbf{y}'}} [\exp(-sr_{\mathbf{y}'\mathbf{x}_2})/\, r_{\mathbf{y}'\mathbf{x}_2}]\, d\mathbf{y}'$$

$$A_{11} = -\sum_{i=1}^{k} \int_{S_i} \left\{ \sum_{i=m+1}^{c} \int_{S_i} [\exp(-sr_{\mathbf{x}_1\mathbf{y}})/\, r_{\mathbf{x}_1\mathbf{y}}] M_{\gamma_i}\,(\mathbf{y}, \mathbf{y}')\, d\mathbf{y} \right\}$$

$$\times [\exp(-sr_{\mathbf{y}'\mathbf{x}_2})/\, r_{\mathbf{y}'\mathbf{x}_2}]\, d\mathbf{y}'$$

$$A_{12} = -\sum_{i=m+1}^{c} \int_{S_i} \left\{ \sum_{i=1}^{k} \int_{S_i} \frac{\partial}{\partial n_{i\mathbf{y}}} [\exp(-sr_{\mathbf{x}_1\mathbf{y}})/\, r_{\mathbf{x}_1\mathbf{y}}] M_{4i}\,(\mathbf{y}, \mathbf{y}')\, d\mathbf{y} \right\}$$

$$\times \left\{ \left(\frac{\partial}{\partial n_{i\mathbf{y}'}} + \gamma_i \right) [\exp(-sr_{\mathbf{y}'\mathbf{x}_2})/\, r_{\mathbf{y}'\mathbf{x}_2}] \right\} d\mathbf{y}'$$

$$A_{13} = \sum_{i=1}^{k} \int_{S_i} \left\{ \sum_{i=c+1}^{n} \int_{S_i} \frac{\partial}{\partial n_{i\mathbf{y}}} [\exp(-sr_{\mathbf{x}_1\mathbf{y}})/r_{\mathbf{x}_1\mathbf{y}}]\, L_{\gamma_i^{-1}}\,(\mathbf{y}, \mathbf{y}')\, d\mathbf{y} \right\}$$

$$\times [\exp(-sr_{\mathbf{y}'\mathbf{x}_2})/\, r_{\mathbf{y}'\mathbf{x}_2}]\, d\mathbf{y}'$$

$$A_{14} = \sum_{i=c+1}^{n} \int_{\Gamma_i} \left\{ \sum_{i=1}^{k} \int_{\Gamma_i} \frac{\partial}{\partial n_{i\mathbf{y}}} [\exp(-sr_{\mathbf{x}_1\mathbf{y}})/r_{\mathbf{x}_1\mathbf{y}}]\, M_{4i}\,(\mathbf{y}, \mathbf{y}')\, d\mathbf{y} \right\}$$

$$\times \left\{ \left(1 + \gamma_i^{-1} \frac{\partial}{\partial n_{i\mathbf{y}'}} \right) [\exp(-sr_{\mathbf{y}'\mathbf{x}_2})/r_{\mathbf{y}'\mathbf{x}_2}] \right\} d\mathbf{y}'$$

$$A_{15} = \sum_{i=k+1}^{m} \int_{S_i} \left\{ \sum_{i=m+1}^{c} \int_{S_i} [\exp(-sr_{\mathbf{x}_1\mathbf{y}})/r_{\mathbf{x}_1\mathbf{y}}]\, M_{\gamma_i}\,(\mathbf{y}, \mathbf{y}')\, d\mathbf{y} \right\}$$

$$\times \frac{\partial}{\partial n_{i\mathbf{y}'}} [\exp(-sr_{\mathbf{y}'\mathbf{x}_2})/\, r_{\mathbf{y}'\mathbf{x}_2}]\, d\mathbf{y}'$$

$$A_{16} = \sum_{i=m+1}^{c} \int_{S_i} \left\{ \sum_{i=k+1}^{m} \int_{S_i} [\exp(-sr_{\mathbf{x}_1\mathbf{y}})/r_{\mathbf{x}_1\mathbf{y}}]\, M_{3i}(\mathbf{y}, \mathbf{y}')\, d\mathbf{y} \right\}$$

$$\times \left\{ \left(\frac{\partial}{\partial n_{i\mathbf{y}'}} + \gamma_i \right) [\exp(-sr_{\mathbf{y}'\mathbf{x}_2})/r_{\mathbf{y}'\mathbf{x}_2}] \right\} d\mathbf{y}'$$

$$A_{17} = -\sum_{i=k+1}^{m} \int_{S_i} \left\{ \sum_{i=c+1}^{n} \int_{S_i} \frac{\partial}{\partial n_{i\mathbf{y}}} [\exp(-sr_{\mathbf{x}_1\mathbf{y}})/r_{\mathbf{x}_1\mathbf{y}}]\, L_{\gamma_i^{-1}}\,(\mathbf{y}, \mathbf{y}')\, d\mathbf{y} \right\}$$

$$\times \frac{\partial}{\partial n_{i\mathbf{y}'}}[\exp(-sr_{\mathbf{y}'\mathbf{x}_2})/r_{\mathbf{y}'\mathbf{x}_2}]\, d\,\mathbf{y}'$$

$$A_{18} = -\sum_{i=c+1}^{n}\int_{S_i}\left\{\sum_{i=k+1}^{m}\int_{S_i}[\exp(-sr_{\mathbf{x}_1\mathbf{y}})/r_{\mathbf{x}_1\mathbf{y}}]M_{3i}\,(\mathbf{y}, \mathbf{y}')\, d\mathbf{y}\right\}$$

$$\times\left\{\left(1+\gamma_i^{-1}\frac{\partial}{\partial n_{i\mathbf{y}'}}\right)[\exp(-sr_{\mathbf{y}'\mathbf{x}_2})/r_{\mathbf{y}'\mathbf{x}_2}]\right\} d\mathbf{y}'$$

$$A_{19} = -\sum_{i=m+1}^{c}\int_{S_i}\left\{\sum_{i=c+1}^{n}\int_{S_i}\frac{\partial}{\partial n_{i\mathbf{y}}}[\exp(-sr_{\mathbf{x}_1\mathbf{y}})/r_{\mathbf{x}_1\mathbf{y}}]\, L^{*}_{\gamma_i^{-1}}\,(\mathbf{y}, \mathbf{y}')\, d\mathbf{y}\right\}$$

$$\times\left\{\left(\frac{\partial}{\partial n_{i\mathbf{y}'}}+\gamma_i\right)[\exp(-sr_{\,\mathbf{y}'\mathbf{x}_2})/r_{\mathbf{y}'\mathbf{x}_2}]\right\} d\mathbf{y}'$$

$$A_{20} = -\sum_{i=c+1}^{n}\left\{\sum_{i=m+1}^{c}\int_{S_i}[\exp(-sr_{\mathbf{x}\,_1\mathbf{y}})/r_{\mathbf{x}_1\mathbf{y}}]\, M_{\gamma_i}\,(\mathbf{y}, \mathbf{y}')\, d\mathbf{y}\right\}$$

$$\times\left\{\left(1+\gamma_i^{-1}\frac{\partial}{\partial n_{i\mathbf{y}'}}\right)[\exp(-sr_{\mathbf{y}'\mathbf{x}_2})/r_{\mathbf{y}'\mathbf{x}_2}]\right\} d\mathbf{y}'$$

where we deduce also that

$$M_{1i}(\mathbf{y}, \mathbf{y}') = \sum_{\nu=0}^{\infty} K_{1i}^{(\nu)}(\mathbf{y}, \mathbf{y}')$$

$$K_{1i}^{(0)}(\mathbf{y}, \mathbf{y}') = \frac{\partial}{\partial n_{i\mathbf{y}'}}[\exp(-sr_{\mathbf{y}'\,\mathbf{y}})/r_{\mathbf{y}'\mathbf{y}}]/2\pi$$

$$M_{2i}(\mathbf{y}, \mathbf{y}') = \sum_{\nu=0}^{\infty} (-1)^{\nu}\, K_{2i}^{(\nu)}(\mathbf{y}, \mathbf{y}')$$

$$K_{2i}^{(0)}(\mathbf{y}, \mathbf{y}') = \frac{\partial}{\partial n_{i\mathbf{y}}}[\exp(-sr_{\mathbf{y}'\mathbf{y}})/r_{\mathbf{y}'\mathbf{y}]/2\pi}$$

$$L_{\gamma_i}(\mathbf{y}, \mathbf{y}') = \sum_{\nu=0}^{\infty}(-1)^{\nu}K_{\gamma_i}^{(\nu)}(\mathbf{y}, \mathbf{y}')$$

$$K_{\gamma_i}^{(0)}(\mathbf{y}, \mathbf{y}') = \left\{\left(\frac{\partial}{\partial n_{i\mathbf{y}}}+\gamma_i\right)[\exp(-sr_{\mathbf{y}'\mathbf{y}})/r_{\mathbf{y}'\mathbf{y}}]\right\}\Big/ 2\pi$$

$$L_{\gamma_i^{-1}}(\mathbf{y}, \mathbf{y}') = \sum_{\nu=0}^{\infty} K_{\gamma_i^{-1}}^{(\nu)}\,(\mathbf{y}, \mathbf{y}')$$

$$K^{(0)}_{\gamma_i^{-1}}(\mathbf{y}, \mathbf{y}') = \left\{\left(\frac{\partial}{\partial n_{i\mathbf{y}'}} + \gamma_i^{-1} \frac{\partial^2}{\partial n_{i\mathbf{y}}\, \partial n_{i\mathbf{y}'}}\right)[\exp(-sr_{\mathbf{y}'\mathbf{y}})/r_{\mathbf{y}'\mathbf{y}}]\right\}\Big/2\pi$$

$$M_{3i}(\mathbf{y}, \mathbf{y}') = \sum_{\nu=0}^{\infty} K^{(\nu)}_{3i}(\mathbf{y}, \mathbf{y}')$$

$$K^{(0)}_{3i}(\mathbf{y}, \mathbf{y}') = \frac{\partial^2}{\partial n_{i\mathbf{y}}\, \partial n_{i\mathbf{y}'}} [\exp(-sr_{\mathbf{y}'\mathbf{y}})/r_{\mathbf{y}'\mathbf{y}}]/2\pi$$

$$M_{4i}(\mathbf{y}, \mathbf{y}') = \sum_{\nu=0}^{\infty} (-1)^{\nu} K^{(\nu)}_{4i}(\mathbf{y}, \mathbf{y}')$$

$$K^{(0)}_{4i}(\mathbf{y}, \mathbf{y}') = [\exp(-sr_{\mathbf{y}'\mathbf{y}})/r_{\mathbf{y}'\mathbf{y}}]/2\pi$$

$$M_{\gamma_i}(\mathbf{y}, \mathbf{y}') = \sum_{\nu=0}^{\infty} (-1)^{\nu}\, {}^{*}K^{(\nu)}_{\gamma_i}(\mathbf{y}, \mathbf{y}')$$

$${}^{*}K^{(0)}_{\gamma_i}(\mathbf{y}, \mathbf{y}') = \left\{\left(\frac{\partial^2}{\partial n_{i\mathbf{y}}\, \partial n_{i\mathbf{y}'}} + \gamma_i \frac{\partial}{\partial n_{i\mathbf{y}}}\right)[\exp(-sr_{\mathbf{y}'\mathbf{y}})/r_{\mathbf{y}'\mathbf{y}}]\right\}\Big/2\pi$$

$$L^{*}_{\gamma_i^{-1}}(\mathbf{y}, \mathbf{y}') = \sum_{\nu=0}^{\infty} {}^{*}K^{(\nu)}_{\gamma_i^{-1}}(\mathbf{y}, \mathbf{y}')$$

$${}^{*}K^{(0)}_{\gamma_i^{-1}}(\mathbf{y}, \mathbf{y}') = \left\{\left(1 + \gamma_i^{-1} \frac{\partial}{\partial n_{i\mathbf{y}}}\right)[\exp(-sr_{\mathbf{y}'\mathbf{y}})/r_{\mathbf{y}'\mathbf{y}}]\right\}\Big/2\pi$$

In these formulas we note, for example, that the $K^{(\nu)}_{1i}(\mathbf{y}, \mathbf{y}')$ are the iterates of $K^{(0)}_{1i}(\mathbf{y}, \mathbf{y}')$. On the basis of (3.13), the function $\bar{\chi}(\mathbf{x}_1, \mathbf{x}_2; s^2)$ will be estimated for large values of s together with small/large impedances γ_i. The case when $\mathbf{x}_1$ and $\mathbf{x}_2$ lie in the neighborhood of the piecewise smooth parts S_i ($i = 1, \ldots, n$) is particularly interesting. For this case, we use the local expansions of the functions:

$$\exp[(-sr_{\mathbf{xy}})/r_{\mathbf{xy}}], \qquad \frac{\partial}{\partial n_{i\mathbf{y}}} [\exp(-sr_{\mathbf{xy}})/r_{\mathbf{xy}}] \tag{3.14}$$

when the distance between $\mathbf{x}$ and $\mathbf{y}$ is small, which are very similar to those obtained in Sections 4 and 5 of ref. 11. Consequently, the local behavior of the kernels

$$K^{(0)}_{1i}(\mathbf{y}, \mathbf{y}'), \quad K^{(0)}_{2i}(\mathbf{y}, \mathbf{y}'), \quad K^{(0)}_{3i}(\mathbf{y}, \mathbf{y}'), \quad K^{(0)}_{4i}(\mathbf{y},\mathbf{y}'), \qquad (i = 1, \ldots, n) \tag{3.15}$$

when the distance between $\mathbf{y}$ and $\mathbf{y}'$ is small follows directly from the knowledge of the local expansions of (3.14). Similarly, when the distance between $\mathbf{y}$ and $\mathbf{y}'$ is small, and for small/large impedances γ_i, the local behavior of the kernels

$$K^{(0)}_{\gamma_i}(\mathbf{y}, \mathbf{y}'), \quad K^{(0)}_{\gamma_i^{-1}}(\mathbf{y}, \mathbf{y}'), \quad {}^*K^{(0)}_{\gamma_i}(\mathbf{y}, \mathbf{y}'), \quad {}^*K^{(0)}_{\gamma_i^{-1}}(\mathbf{y}, \mathbf{y}') \qquad (i = 1, \ldots, n) \tag{3.16}$$

follows directly from the knowledge of the local expansions of (3.14).

Definition 3.1. If ξ_1 and ξ_2 are points in the half-part $\xi^3 > 0$, then we define

$$\rho_{12} = [(\xi_1^1 - \xi_2^1)^2 + (\xi_1^2 - \xi_2^2)^2 + (\xi_1^3 + \xi_2^3)^2]^{1/2}$$

An e^λ $(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2; s)$ function is defined for points $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ belongs to sufficiently small domains $D(I_i)$ $(i = 1, \ldots, n)$ except when $\boldsymbol{\xi}_1 = \boldsymbol{\xi}_2 \in I_i$, and λ is called the degree of this function. For every positive integer $\wedge$, it has the following local expansion [8, 11, 14]:

$$e^\lambda(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2; s) = \Sigma^* f(\xi_1^1), \xi_1^2)(\xi_1^3)^{p_1}(\xi_2^3)^{p_2} \left(\frac{\partial}{\partial \xi_1^1}\right)^{l_1} \left(\frac{\partial}{\partial \xi_1^2}\right)^{l_2} \left(\frac{\partial}{\partial \xi_1^3}\right)^{l_3}$$

$$\times \frac{\exp(-s\rho_{12})}{\rho_{12}} + R^\wedge(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2; s) \tag{3.17}$$

where Σ^* denotes a sum of a finite number of terms in which $f(\xi_1^1, \xi_1^2)$ is an infinitely differentiable function. In this expansion p_1, p_2, l_1, l_2, l_3 are integers, where $p_1 \geq 0$, $p_2 \geq 0$, $l_1 \geq 0$, $l_2 \geq 0$, and $\lambda = \min(p_1 + p_2 - q)$, where $q = l_1 + l_2 + l_3$, and the minimum is taken over all terms which occur in the summation Σ^*. The remainder $R^\wedge(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2; s)$ has continuous derivatives of order $d \leq \wedge$ satisfying

$$D^d R^\wedge(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2; s) = O\,\{s^{-\wedge} \exp(-As\rho_{12})\} \qquad \text{as} \quad s \to \infty \tag{3.18}$$

where A is a positive constant.

Thus, using methods similar to those in Sections 6–10 of Zayed [11], we can show that the functions (3.14) are e^λ functions with degrees $\lambda = -1, -2$, respectively. Consequently, the functions (3.15) are e^λ functions with degrees $\lambda = 0, 0, -1, 1$, respectively, while for small/large γ_i, the functions (3.16) are e^λ functions with degrees $\lambda = 0, 0, -1, 1$, respectively.

Definition 3.2. If $\mathbf{x}_1$ and $\mathbf{x}_2$ are points in large domains $\Omega + S_i$ $(i = 1, \ldots, n)$, then we define

$$r_{12} = \min_y (r_{\mathbf{x}_1\mathbf{y}} + r_{\mathbf{x}_2\mathbf{y}}) \qquad \text{if} \quad \mathbf{y} \in S_i \ (i = 1, \ldots, k)$$

$$R_{12} = \min_y (r_{\mathbf{x}_1\mathbf{y}} + r_{\mathbf{x}_2\mathbf{y}}) \qquad \text{if} \quad \mathbf{y} \in S_i \ (i = k + 1, \ldots, m)$$

$$r_{12}^* = \min_y (r_{\mathbf{x}_1\mathbf{y}} + r_{\mathbf{x}_2\mathbf{y}}) \qquad \text{if} \quad \mathbf{y} \in S_i \ (i = m+1, \ldots, c)$$

$$R_{12}^* = \min_y (r_{\mathbf{x}_1\mathbf{y}} + r_{\mathbf{x}_2\mathbf{y}}) \qquad \text{if} \quad \mathbf{y} \in S_i (i = c+1, \ldots, n)$$

An $E^\lambda(\mathbf{x}_1, \mathbf{x}_2; s)$ function is defined and infinitely differentiable with respect to $\mathbf{x}_1$ and $\mathbf{x}_2$ when these points belong to large domains $\Omega + S_i$ ($i = 1, \ldots, n$) except when $\mathbf{x}_1 = \mathbf{x}_2 \in S_i$. Thus, the E^λ function has a similar local expansion to the e^λ function [8, 11, 14].

With the help of Section 8 in ref. 11, it is easily seen that the formula (3.13) is an $E^{-2}(\mathbf{x}_1, \mathbf{x}_2; s)$ function and consequently we have the estimate

$$\overline{G}(\mathbf{x}_1, \mathbf{x}_2; s^2) = \sum_{i=1}^{k} O\{r_{12}^{-2} \exp(-A_i s r_{12})\} + \sum_{i=k+1}^{m} O\{R_{12}^{-2} \exp(-A_i s R_{12})\}$$
$$+ \sum_{i=m+1}^{c} O\,\{r_{12}^{*\,-2} \exp(-A_i s r_{12}^*)\} + \sum_{i=c+1}^{n} O\{R_{12}^{*\,-2} \exp(-A_i s R_{12}^*)\} \tag{3.19}$$

which is valid for $s \to \infty$ and for small/large impedances γ_i, where A_i are positive constants. The estimate (3.19) shows that $\overline{G}(\mathbf{x}_1, \mathbf{x}_2; s^2)$ is exponentially small for s large. This proves that $\overline{G}\,(\mathbf{x}_1, \mathbf{x}_2; s^2)$ converges for $s \to \infty$. With reference to Section 10 in Zayed [11], if the e^λ expansions of the functions (3.14)–(3.16) are introduced into (3.13) and if we use formulas similar to (6.4) and (6.9) of Section 6 in ref. 11, we obtain the following behavior of $\overline{\chi}(\mathbf{x}_1, \mathbf{x}_2; s^2)$ when r_{12}, R_{12}, r_{12}^*, and R_{12}^* are small, which is valid for $s \to \infty$ and for small/large γ_i:

$$\overline{\chi}(\mathbf{x}_1, \mathbf{x}_2; s^2) = \sum_{i=1}^{n} \overline{\chi}_i(\mathbf{x}_1, \mathbf{x}_2; s^2) \tag{3.20}$$

where (a) if $\mathbf{x}_1$ and $\mathbf{x}_2$ belong to sufficiently small domains $D(I_i)$ ($i = 1, \ldots, k$),

$$\overline{\chi}_i(\mathbf{x}_1, \mathbf{x}_2; s^2) = -\frac{\exp(-s\rho_{12})}{8\pi\rho_{12}} + O\{\rho_{12}^{-1} \exp(-A_i s\, \rho_{12})\} \tag{3.21}$$

(b) if $\mathbf{x}_1$ and $\mathbf{x}_2$ belong to sufficiently small domains $D(I_i)$ ($i = k+1, \ldots, m$),

$$\overline{\chi}_i(\mathbf{x}_1, \mathbf{x}_2; s^2) = \frac{\exp(-s\rho_{12})}{8\pi\rho_{12}} + O\{\rho_{12}^{-1} \exp(-A_i s\rho_{12}\} \tag{3.22}$$

(c) if $\mathbf{x}_1$ and $\mathbf{x}_2$ belong to sufficiently small domains $D(I_i)$ ($i = m+1, \ldots, c$),

$$\overline{\chi}_i(\mathbf{x}_1, \mathbf{x}_2; s^2) = \frac{1}{8\pi}\left\{1 - \gamma_i\left(\frac{\partial}{\partial \xi_1^3}\right)^{-1}\right\}\frac{\exp(-s\rho_{12})}{\rho_{12}} + O\{\rho_{12}^{-1}\exp(-A_i s\rho_{12}\} \tag{3.23}$$

(d) if $\mathbf{x}_1$ and $\mathbf{x}_2$ belong to sufficiently small domains $D(I_i)$ $(i = c + 1, \ldots, n)$,

$$\overline{\chi}_i(\mathbf{x}_1, \mathbf{x}_2; s^2) = -\frac{1}{8\pi}\left\{1 - \gamma_i^{-1}\left(\frac{\partial}{\partial \xi_1^3}\right)\right\}\frac{\exp(-s\rho_{12})}{\rho_{12}} + O\{\rho_{12}^{-1}\exp(-A_i s\rho_{12}\} \tag{3.24}$$

When $r_{12} \geq \delta_i > 0$ $(i = 1, \ldots, k)$, $R_{12} \geq \delta_i > 0$ $(i = k + 1, \ldots, m)$, $r_{12}^* \geq \delta_i > 0$ $(i = m + 1, \ldots, c)$, and $R_{12}^* \geq \delta_i > 0$ $(i = c + 1, \ldots, n)$, the function $\overline{\chi}(\mathbf{x}_1, \mathbf{x}_2; s^2)$ is of the order $O\{\exp(-Bs)\}$ as $s \to \infty$, $B = \text{const} > 0$. Thus, since

$$\lim_{r_{12}\to 0}\frac{r_{12}}{\rho_{12}} = \lim_{R_{12}\to 0}\frac{R_{12}}{\rho_{12}} = \lim_{r_{12}^*\to 0}\frac{r_{12}^*}{\rho_{12}} = \lim_{R_{12}^*\to 0}\frac{R_{12}^*}{\rho_{12}} = 1$$

[8, 11], then we have the asymptotic formulas (3.21)–(3.24) with ρ_{12} in the small domains $D(I_i)$ $(i = 1, \ldots, n)$ are replaced by r_{12}, R_{12}, r_{12}^*, R_{12}^* in the large domains $\Omega + S_i$ $(i = 1, \ldots, n)$, respectively.

Since for $\xi^3 \geq h_i > 0$ $(i = 1, \ldots, n)$ the functions $\overline{\chi}_i(\mathbf{x}, \mathbf{x}; s^2)$ are of order $O\{\exp(-2A_i s h_i\}$, the integral over Ω of the function $\overline{\chi}(\mathbf{x}, \mathbf{x}; s^2)$ can be approximated in the following way [see (3.10)]:

$$\overline{R}(s^2) = \sum_{i=1}^{n}\int_{S_i}\int_{\xi^3=0}^{h_i}\overline{\chi}_i(\mathbf{x}, \mathbf{x}; s^2)\,\{1 - 2\xi^3 H + (\xi^3)^2 N\}\,d\xi^3\,dS_i + \sum_{i=1}^{n} O\{\exp(-2A_i s h_i\} \quad \text{as} \quad s \to \infty \tag{3.25}$$

If the e^λ expansions of $\overline{\chi}_i(\mathbf{x}, \mathbf{x}; s^2)$ are introduced into (3.25), then with the help of the formula (11.2) of Section 11 in Zayed [11], we deduce after inverting Laplace transforms and using (3.6), that the asymptotic expansion (2.1) has been constructed, and the proof of Theorem 2.1 follows. ■

4. AN APPLICATION OF THE INVERSE PROBLEM FOR AN IDEAL GAS

With reference ref. 3, we are interested in examining how the thermodynamic properties of an ideal gas are influenced by the geometry of its container.

The thermodynamic properties of an ideal gas can be extracted from the partition function

$$Z = \frac{z^r}{r!} \tag{4.1}$$

where r is the number of particles and z is given by

$$z = \sum_{\nu} \exp(-\beta E_{\nu}) \tag{4.2}$$

where $\beta = (K_B T)^{-1}$, K_B is Boltzmann's constant, and T is the absolute temperature. The eigenvalues (energy levels) of one particle E_ν are obtained from the stationary states $\Psi(\mathbf{x}, t) = u(x)\exp(-iEt/h)$ of the time-dependent Schrodinger equation

$$-\frac{h^2}{2M}\nabla^2\psi + V\ \mathbf{x})\psi = ih\frac{\partial\psi}{\partial t} \tag{4.3}$$

with $V = 0$, where M is the mass of the particles and h is the Planck constant. Thus $u(\mathbf{x})$ obey the Helmholtz equation (1.1) with $\mu = 2ME/h^2$ and with Dirichlet, Neumann, and Robin boundary conditions (1.8). Note that the trace of the heat kernel $\Theta(t)$ given by (1.4) of Section 1 formally is the same as the one-particle partition function $z(\beta)$ given by (4.2).

The purpose of this section is to use our main result (2.1) of Section 2 to derive a general expression for the corrections to the thermodynamic quantities, particularly the energy for an ideal gas due to a large but finite container volume.

Following Section 2, we can obtain information about the shape of the domain by studying the asymptotic value of the sum (4.2) with $\beta \to 0$ (i.e., $T \to \infty$, the ideal gas case). Noting that the eigenvalue problem of the Schrödinger equation is the same as the eigenvalue of the wave equation with Dirichlet, Neumann, and Robin conditions, we can use directly our result (2.1) replacing t by $(h^2/2M)\beta$. Let us now consider the general partition function (4.1). Using directly the result (2.1) with $\beta \to 0$, we find that equation (4.2) gives

$$\begin{aligned} z(\beta) = {} & \left(\frac{2M}{h^2}\right)^{3/2}\frac{a_1}{\beta^{3/2}} + \left(\frac{2M}{h^2}\right)\frac{a_2}{\beta} + \left(\frac{2M}{h^2}\right)^{1/2}\frac{a_3}{\beta^{1/2}} + a_4 \\ & + \left(\frac{h^2}{2M}\right)^{1/2} a_5\beta^{1/2} + O(\beta) \end{aligned} \tag{4.4}$$

We set out to apply this formula to thermodynamic quantities such as the internal energy $U = -((\partial/\partial\beta)\ln Z)_{\nu,r}$, the pressure $P = \beta^{-1}((\partial/\partial V)\ln Z)_{T,r}$, and the specific heat $C = (\partial U/\partial T)_{v,r}$ among others. In the case of the internal energy, we obtain

$$U = -r\frac{\partial}{\partial\beta}\ln\left\{\left(\frac{2M}{h^2}\right)^{3/2}\frac{a_1}{\beta^{3/2}} + \left(\frac{2M}{h^2}\right)\frac{a_2}{\beta} + \left(\frac{2M}{h^2}\right)^{1/2}\frac{a_3}{\beta^{1/2}} + a_4 + \left(\frac{h^2}{2M}\right)^{1/2} a_5\beta^{1/2} + O(\beta)\right\} \tag{4.5}$$

Now, differentiating, expanding in powers of $\beta = (K_B T)^{-1}$, and using the definition of the thermal wavelength $\Lambda(T) = (2\pi h^2/MK_B T)^{1/2}$, we obtain

$$\begin{aligned} U(T) = \frac{3}{2} r K_B T\Bigg\{ & 1 - \frac{a_2}{6a_1\sqrt{\pi}}\Lambda(T) + \frac{1}{12\pi}\left[\left(\frac{a_2}{a_1}\right)^2 - \frac{2a_3}{a_1}\right]\Lambda^2(T) \\ & - \frac{1}{24\pi^{3/2}}\left[\left(\frac{a_2}{a_1}\right)^3 - \frac{3a_2a_3}{a_1^2} + \frac{3a_4}{a_1}\right]\Lambda^3(T) \\ & + \frac{1}{48\pi^2}\left[\left(\frac{a_2}{a_1}\right)^4 + \frac{4a_2a_4}{a_1^2} - \frac{4a_2^2a_3}{a_1^3} + 2\left(\frac{a_3}{a_1}\right)^2 - \frac{4a_5}{a_1}\right]\Lambda^4(T) + O(\Lambda^5(T))\Bigg\} \\ & \text{as} \quad T \to \infty \end{aligned} \tag{4.6}$$

Similar expressions hold for the pressure and the specific heat.

Thus, we have investigated the influence of the finite container Ω on the thermodynamic quantities of an ideal gas. The calculations are based on the asymptotic expansion formula (2.1) of the spectrum of the Laplacian. The energy is obtained by the formula (4.6) as an expansion in powers of the thermal wavelength, whose coefficients depend on some geometric properties of the container Ω. Thus, in principal, an ideal gas could feel some aspects of the shape of its container.

We close this section with the remark that Gutierrez and Yanez [3] have recently constructed a formula similar to our formula (4.6) but for a simply connected bounded domain with Dirichlet condition by using Waechter's formula (1.6) [10]. Of course, according to Gordon *et al.* [1], there are domains where, although different in shape, the thermodynamic properties of an ideal gas will be exactly the same, independent of the order of the approximation in (4.6). In this sense, an ideal gas cannot feel the shape of its container, although it can feel some geometric properties.

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